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Stability of Heavy Circular Arches with Hinged Ends

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THE problem of buckling of nonshallow arches subjected to static loads can be subdivided into two classes: 1) arches whose undeformed centroidal line coincides with the funicular curve of the applied loading, and whose instability is preceded by small displacements; and 2) arches that buckle by sidesway or snap-through at large deflections. The first class has been investigated rather extensively (for a bibliography of fundamental problems see Refs. 1 and 2). Its problems are of the Euler type, amenable to comparatively simple analysis.

The problems of the second class are highly nonlinear and present great computational difficulties. Nevertheless, a considerable number of papers has appeared recently dealing with the stability of arches subjected to concentrated loads.[3–11] The problem of distributed loads on nonshallow arches with large deflections has not been investigated analytically, and it is the purpose of this Note to initiate research in that direction. Of practical interest are behaviors of arches with circular, parabolic, and catenarian centroidal curves subjected to dead weight, uniform and piecewise uniform vertical load, and combinations of various loads. However, the present investigation will be restricted to two-hinged circular arches of constant cross section buckling under their own weight.

The theory of plane curved beams and arches with very large deflections has been presented in several publications, e.g., Refs. 8, 12, and 13. For that reason, only a summary of necessary equations is presented herein.

Assuming an inextensible centroidal curve—an assumption that is commonly made for nonshallow arches—we can write the differential equations of equilibrium in the form

$$M' = aQ \tag{1a}$$

$$aN' - (1 + \beta')M' = -a^2 w \sin(\phi + \beta - \alpha) \tag{1b}$$

$$M'' + a(1 + \beta')N = -a^2 w \cos(\phi + \beta - \alpha) \tag{1c}$$

where M is the bending couple; Q the shearing force; N the normal tensile force; w the dead weight of the arch rib, per unit length; a the constant radius of the undeformed centroidal curve; β the angle of rotation of a line element of the centroidal curve; ϕ the angle measured from a reference radius; and primes indicate differentiation with respect to ϕ (see Fig. 1). The bending couple M is related to β' by

$$M = -(EI/a)\beta' \tag{2}$$

according to the hypothesis of plane cross sections. Herein, E is the modulus of elasticity and I the moment of inertia of the cross-sectional area of arch rib.

The geometrical relations are given by

$$v' + u = a \sin\beta, \qquad u' - v = a(\cos\beta - 1) \tag{3}$$

where u and v are the tangential and normal displacement components of the centroidal curve (see Fig. 1).

The boundary conditions are

$$u = v = M = 0 \tag{4}$$

at the two hinges.

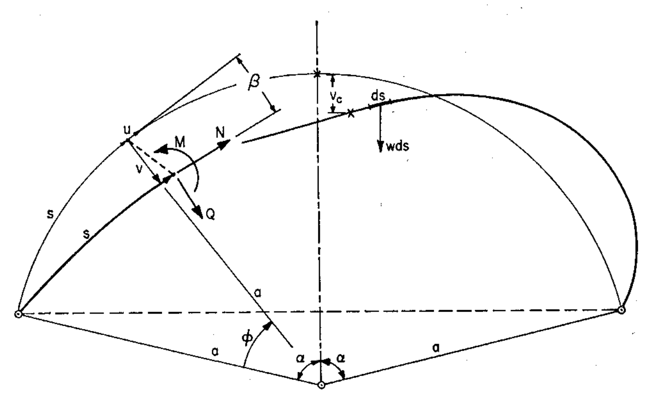

Fig. 1 Deformed and undeformed centroidal curve of the arch.

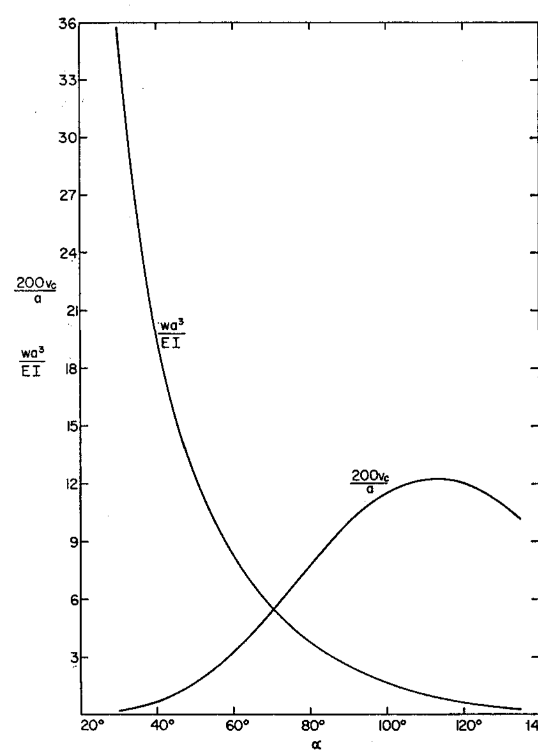

Fig. 2 Variation of the critical load w and critical vertical displacement v_c of the crown with the subtending angle 2α.

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The foregoing boundary-value problem was solved with the aid of a digital computer. It was found that the arch buckles by sidesway (see Fig. 1), displaying bifurcations in the load-deflection curves. The calculated values of the critical load w and the critical vertical displacement v_c of the crown are presented in Fig. 2 for different subtending angles 2α. As seen from this figure, the larger the arch rise, the smaller is the intensity of the critical distributed load for given values of the initial radius a and flexural rigidity EI. Shallow arches have not been considered, as the inextensible theory yields accurate results for deep arches only.

Of some interest should be the comparison of the total critical weight $2\alpha aw$ with the critical concentrated downward load P at the crown of the arch. For example,

$$2\alpha aw = 24.4\ EI/a^2, \qquad P = 15.3\ EI/a^2 \qquad \text{for } \alpha = 45°$$

$$2\alpha aw = 7.85\ EI/a^2, \qquad P = 5.86\ EI/a^2 \qquad \text{for } \alpha = 90°$$

$$2\alpha aw = 1.17\ EI/a^2, \qquad P = 0.89\ EI/a^2 \qquad \text{for } \alpha = 135°$$

where the critical values for P are taken from Refs. 9 and 11. (Remark: The critical values of $P = 6.17\ EI/a^2$ and $12.74\ EI/a^2$ for $\alpha = 90°$ and $53.1301°$, respectively, as calculated by a finite element method in Refs. 14 and 15, do not agree well with our values of $5.86\ EI/a^2$ and $13.00\ EI/a^2$ which have been calculated by means of an exact theory.)

The foregoing comparison of buckling loads may be of some utility in the interpretation of test results for arches subjected to concentrated loads, since it is difficult to eliminate the effect of the dead weight of the arch tested.

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Vibrations of Pressurized Orthotropic Shells

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Introduction

THE purpose of this short Note is to extend for the sake of completeness some previously published results on the vibration of pressurized orthotropic cylindrical membranes.[1] In that paper the effects of a) internal pressurization, b) the variation of elastic constants, and c) the deletion of the inplane inertia, upon the free and forced vibrations of membrane shells was examined. The present analysis will display the effects of these physically important parameters upon the free vibrations of shells where the bending terms are included.

Analysis

The formulation is based on the nonlinear strain-displacement relations used in the theory of shallow shells. With u^*, v^*, w^* being the dimensional displacements they take the form

$$\begin{aligned} \epsilon_{xx} &= \partial u^*/\partial x + \tfrac{1}{2}(\partial w^*/\partial x)^2 - z\partial^2 w^*/\partial x^2 \\ \epsilon_{yy} &= \partial v^*/\partial y - w^*/R + \tfrac{1}{2}(\partial w^*/\partial y)^2 - z\partial^2 w^*/\partial y^2 \\ \gamma_{xy} &= \partial u^*/\partial y + \partial v^*/\partial x + (\partial w^*/\partial x)\partial w^*/\partial y - 2z\partial^2 w^*/\partial x\partial y \end{aligned} \tag{1}$$

These kinematic equations are used in conjunction with the linear orthotropic relations for a conservative material in a state of plane stress

$$\begin{aligned} \tau_{xx} &= C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} \\ \tau_{yy} &= C_{12}\epsilon_{xx} + C_{22}\epsilon_{yy},\ \tau_{xy} = G_{12}\gamma_{xy} \end{aligned} \tag{2}$$

and Hamilton's principle to derive the following set of equilibrium equations:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \frac{\partial^2 u^*}{\partial t^2},\quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = \frac{\partial^2 v^*}{\partial t^2} \tag{3a}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w^*}{\partial x} + N_{xy}\frac{\partial w^*}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w^*}{\partial x} + N_{yy}\frac{\partial w^*}{\partial y}\right) + \frac{1}{R}N_{yy} = \frac{\partial^2 w^*}{\partial t^2} \tag{3b}$$

Here the N_{ij} and M_{ij} are the usual stress and moment result-

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